Production and Advertising in a Dynamic Hotelling Monopoly[1]

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Abstract

I characterise the dynamics of capacity accumulation and investment in advertising in a spatial monopoly model, contrasting the socially optimal behaviour of a benevolent planner against the behaviour of a profit-maximising monopolist. I show that, in steady state, the monopolist always distorts both kinds of investment as compared to the social optimum, except in a situation where the Ramsey equilibrium prevails under both regimes.

Keywords: capital accumulation, advertising, steady state, saddle point

JEL Classification: D24, L12, M37

1 Introduction

The analysis of dynamic monopoly dates back to Evans (1924) and Tintner (1937), who analysed the pricing behaviour of a …rm subject to a U-shaped variable cost curve.[1] The analysis of intertemporal capital accumulation appeared later on (see Eisner and Strotz, 1963, inter alia).

The existing literature investigates several features of monopoly markets, in particular several forms of discrimination, either through (intertemporal) pricing (see Stokey, 1981; Bulow, 1982; Gul, Sonnenschein and Wilson, 1986) or through product proliferation (see Mussa and Rosen, 1978; Maskin and Riley, 1984; Gabszewicz, Shaked, Sutton and Thisse, 1986; Bonanno, 1987).

Another dynamic tool which has received a considerable amount of attention is advertising, ever since Vidale and Wolfe (1957) and Nerlove and Arrow (1962).[2] A taxonomy introduced by Sethi (1977) distinguishes between advertising capital models and sales-advertising response models. The …rst category considers advertising as an investment in a stock of goodwill, à la Nerlove-Arrow. The second category gathers models where there exists a direct relationship between the rate of change in sales and advertising, à la Vidale-Wolfe.

In this paper, I propose a monopoly model where the …rm locates the product in a spatial market representing the space of consumer preferences, as in Hotelling (1929). The volume of sales at any point in time depends upon consumers' reservation price (or, equivalently, willingness to pay for the product), and the …rm may invest in an advertising campaign in order to increase consumers' reservation price, in the Nerlove-Arrow vein. Moreover, supplying the market involves building up productive capacity, and this may

[1] See Chiang (1992) for a recent exposition of the original model by Evans, as well as later developments.

[2] For exhaustive surveys, see Sethi (1977); Jørgensen (1982); Feichtinger and Jørgensen (1983); Erickson (1991); Feichtinger, Hartl and Sethi (1994). For duopoly models with dynamic pricing and advertising, see in particular Leitmann and Schmitendorf (1978), and Feichtinger (1983). For more recent developments, see the Proceedings of IX ISDG (2000).

interact with the advertising effort.

I characterise the dynamics of capacity accumulation and investment in advertising first in isolation and then jointly, contrasting the socially optimal behaviour of a benevolent planner against the behaviour of a profit-maximising monopolist. I show that, in steady state, the monopolist always distorts both kinds of investment as compared to the social optimum, except in a situation where a Ramsey-like equilibrium prevails under both regimes. That is:

- There exists a long run equilibrium where the firm operates with a productive capacity which is only driven by demand conditions. In this situation, the profit-seeking monopolist invests too little resources in both advertising and production, as compared to a benevolent planner.
- There exists a long run equilibrium where the firm operates with a productive capacity which is only driven by considerations concerning time discounting and depreciation. In this situation, the behaviour of the firm in steady state is the same irrespective of the regime being considered. Accordingly, monopoly distortions disappear.

The remainder of the paper is structured as follows. The setup is laid out in section 2. The capital accumulation problem is investigated in section 3. Advertising is described in section 4. The interaction between capacity accumulation and advertising is analysed in section 5.

2 The model

The setup shares its basic features with d'Aspremont, Gabszewicz and Thisse (1979). I consider a market for horizontally differentiated products where consumers are uniformly distributed with unit density along the unit interval $[0,1]$. Let the market exist over $t \in [0,\infty)$. The market is served by a single firm selling a single good located at $\ell(t) \in [0;1]$: Product location is costless.[3] The generic consumer located at $a(t) \in [0;1]$ buys one unit of the good, if net surplus from purchase is non-negative:

$$U(t) = s(t) - p(t) - [\ell(t) - a(t)]^2 \geq 0; \qquad i = 1;2; \tag{1}$$

where $p(t)$ is the firm's mill price, and $s(t)$ is gross consumer surplus, that is, the reservation price that a generic consumer is willing to pay for the good. Therefore, $s(t)$ can be considered as a preference parameter which, together with the disutility of transportation, yields a measure of consumers' taste for the good. The mill price is such that marginal consumers at distance $|\ell(t) - a(t)|$ from the store enjoy zero surplus, that is,

$$p(t) = s(t) - [\ell(t) - a(t)]^2 : \tag{2}$$

Observe that, in line of principle, it could be possible to have $q(t) = a(t)$ (if $a(t) > \ell(t)$) or $q(t) = 1 - a(t)$ (if $a(t) < \ell(t)$). However, this situation would be clearly suboptimal for the monopolist, in that he could gain by relocating the product costlessly until demand becomes symmetric around $\ell(t)$. Therefore, the choice of location can be solved once and for all at $t = 0$ by setting $\ell(t) = 1/2$: The same location is also optimal for a benevolent social planner aiming at the maximisation of total surplus.[4] The demand $q(t)$ is then easily defined as the interval $[1 - a(t); a(t)]$; i.e., $q(t) = 2a(t) - 1 \in [0;1]$; provided that

$$a(t) \in (\ell(t);1] \text{ and } 1 - a(t) \in [0;\ell(t)) \text{ if } a(t) \in (\ell(t);1] : \tag{3}$$

I assume that the firm operates at constant marginal production cost, and, for the sake of simplicity, I normalise it to zero. Accordingly, instantaneous

[3] The monopolist's R&D investment for product innovation is investigated in a companion paper (Lambertini, 2000). R&D for product innovation in duopoly is analysed by Harter (1993).

[4] For the sake of brevity, the proof of these claims is omitted, as it is well known from the existing literature (see Bonanno, 1987; and Lambertini, 1995, inter alia).

revenues are:

$$R(t) = p(t)q(t) = \left[s(t) - \left(\frac{1}{2} - a(t) \right)^2 \right] [2a(t) - 1] \tag{4}$$

Instantaneous consumer surplus is:

$$CS(t) = \int_{1-a}^{a} \left[s - p(t) - \left(\frac{1}{2} - m \right)^2 \right] dm = \frac{(2a - 1)^3}{6} \tag{5}$$

Therefore, instantaneous social welfare amounts to

$$SW(t) = R(t) + CS(t) = \frac{(2a(t) - 1)\,[12s - 1 + 4a(t)\,(1 - a(t))]}{12} \tag{6}$$

In the remainder, I will consider, first in isolation and then jointly, the following scenarios:

[1] Production requires physical capital k, accumulating over time to create capacity. At any t, the output level is $y(t) = f(k(t))$, with $f' \equiv \partial f(k(t))/\partial k(t) > 0$ and $f'' \equiv \partial^2 f(k(t))/\partial k(t)^2 < 0$.

A reasonable assumption is that $q(t) \leq y(t)$; that is, the level of sales is at most equal to the quantity produced. Excess output is reintroduced into the production process yielding accumulation of capacity according to the following process:

$$\frac{\partial k(t)}{\partial t} = f(k(t)) - q(t) - \delta k(t)\,; \tag{7}$$

where δ denotes the rate of depreciation of capital. The cost of capital is represented by the opportunity cost of intertemporal relocation of unsold output.[5] Let the initial state be $k(0) = 0$. When capacity accumulation is used in isolation, the willingness to pay $s(t)$ remains constant over t at s_0:

[5] The adoption of technology (7) in the economic literature dates back to Ramsey (1928). For an application to a dynamic oligopoly model with either price or quantity competition, see Cellini and Lambertini (1998).

[2] The monopolist may invest in an advertising campaign over time, aimed at increasing consumers' tastes, i.e., the willingness to pay $s(t)$; according to the following dynamics:

$$\frac{\partial s(t)}{\partial t} = b\sqrt{x(t)} - \delta s(t) \; ; \; b > 0 \; ; \tag{8}$$

where $x(t)$ is the instantaneous investment in advertising, and δ is the constant depreciation rate affecting consumers' willingness to pay.[6] Let the initial state be $s(0) = s_0$. When the advertising technology is used in isolation, production is assumed to be completely costless.

Scenarios [1] and [2] give rise to three cases, namely, (i) capital accumulation for production, given the reservation price s_0; (ii) investment in advertising, with production being carried out at no cost; (iii) capital accumulation for production plus investment in advertising. In all the three cases under consideration, I will first investigate the behaviour of a social planner running the firm so as to maximise net discounted welfare, and then contrast the behaviour of a profit-seeking monopolist against the social planning benchmark.

3 Capital accumulation for production

3.1 Capital accumulation under social planning

In scenario 1, the objective of a benevolent social planner is

$$\max_{a(t)} \int_0^\infty e^{-\rho t} SW(t)dt = \tag{9}$$

$$= \int_0^\infty e^{-\rho t} \frac{(2a(t) - 1)\left[12 s_0 - 1 + 4a(t)(1 - a(t))\right]}{12} dt$$

$$s.t. \quad \frac{\partial k(t)}{\partial t} = f(k(t)) - q(t) - \delta k(t) \tag{10}$$

[6] The choice of (8) is justified by several studies showing the existence of decreasing returns to scale in advertising activities (see Chintagunta and Vilcassim, 1992; Erickson, 1992; see also Feichtinger, Hartl and Sethi, 1994).

where ρ denotes time discounting. In choosing the optimal location of the marginal consumer(s) at any t, the planner indeed maximises discounted social welfare w.r.t. output (or, alternatively, price). The corresponding Hamiltonian function is:

$$
\begin{aligned}
H(t) \;=\; & e^{-\rho t} \cdot \left\{ \frac{(2a(t) - 1)\left[12 s_0 - 1 + 4a(t)(1 - a(t))\right]}{12} + \right. \\
& \left. \lambda(t)\left[f(k(t)) - 2a(t) + 1 - \delta k(t)\right] \right\},
\end{aligned} \tag{11}
$$

where $\lambda(t) = \beta(t)e^{\rho t}$, and $\beta(t)$ is the co-state variable associated to $k(t)$. The first order conditions[7] for a path to be optimal are:

$$
\frac{\partial H(t)}{\partial a(t)} = -2\left[a(t)\right]^2 + 2a(t) + 2s_0 - \frac{1}{2} - 2\lambda(t) = 0\,; \tag{12}
$$

$$
-\frac{\partial H(t)}{\partial k(t)} = \frac{\partial \beta(t)}{\partial t} \;\Rightarrow\; \frac{\partial \lambda(t)}{\partial t} = \left[\rho + \delta - f'(k(t))\right]\lambda(t)\,; \tag{13}
$$

$$
\lim_{t \to \infty} \beta(t) \cdot k(t) = 0\,. \tag{14}
$$

From (12), I obtain[8]

$$
a(t) = \frac{1}{2} + \sqrt{s_0 - \lambda(t)}\,. \tag{15}
$$

In combination with (2) and $\ell(t) = 1/2$, (15) establishes the following result:

Lemma 1 Under social planning, the market price of the final good and the shadow price of capital coincide, i.e., $p(t) = \lambda(t)$.

Expression (15) can be differentiated w.r.t. time to get

$$
\frac{da(t)}{dt} = \frac{-d\lambda(t)/dt}{2\sqrt{s_0 - \lambda(t)}}\,. \tag{16}
$$

[7] Second order conditions are met throughout the paper. They are omitted for the sake of brevity.

[8] Recall that I consider that case where $a(t) \in (1/2, 1]$, so that the other solution to (12) can be excluded.

Using (13), the expression in (16) can be rewritten as follows:

$$\frac{da(t)}{dt} = -\frac{[\rho + \delta - f'(k(t))]\,\lambda(t)}{2\sqrt{s_0 - \lambda(t)}}\,. \tag{17}$$

Moreover, using $\lambda(t) = a(t)\,[1 - a(t)] + s_0 - 1/4$, (17) simplifies as:

$$\frac{da(t)}{dt} = -\frac{\left[a(t) - (a(t))^2 + s_0 - 1/4\right][\rho + \delta - f'(k(t))]}{2\,[a(t) - 1]}\,. \tag{18}$$

Therefore,

$$\operatorname{sign}\left\{\frac{da(t)}{dt}\right\} = \operatorname{sign}\left\{\left[a(t) - (a(t))^2 + s_0 - 1/4\right][f'(k(t)) - \rho - \delta]\right\}. \tag{19}$$

The r.h.s. expression in (19) is zero at

$$f'(k(t)) = \rho + \delta\,; \tag{20}$$

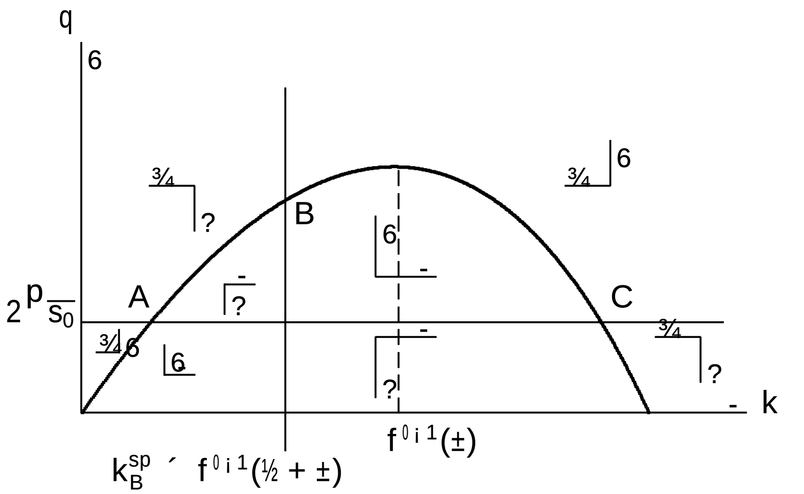
$$a(t) = \frac{1}{2} \pm \sqrt{s_0}\,. \tag{21}$$

The critical point (20) denotes the situation where the marginal product of capital is just sufficient to cover discounting and depreciation. The smaller solution in (21) can be disregarded on the basis of the assumption that $a(t) \in (1/2\,;\,1]$. Therefore, the long run equilibrium output is either $q^{sp}(t) = 2\sqrt{s_0}$, where superscript sp stands for social planning, or the quantity corresponding to a capacity $f'^{-1}(\rho + \delta)$. It is also worth noting that, in correspondence of $a(t) = 1/2 + \sqrt{s_0}$; we have (i) $q^{sp}(t) \leq 1$ for all $s_0 \leq 1/4$, and (ii) $p^{sp}(t) = 0$; i.e., the planner sells the product at marginal cost.

I am now able to draw a phase diagram in the space $\{k; q\}$, in order to characterise the steady state equilibrium (to ease the exposition, the indication of time is dropped in the remainder of the discussion). The locus $\dot{q} \equiv dq/dt = 0$ is given by $q^{sp} = 2\sqrt{s_0}$ and $f'(k) = \rho + \delta$ in figure 1. It is easily shown that the horizontal locus $q^{sp} = 2\sqrt{s_0}$ denotes the usual equilibrium solution we are well accustomed with from the static model. The two loci partition the space $\{k; q\}$ into four regions, where the dynamics of q is determined by (18) with $a = (q + 1)/2$, as summarised by the vertical arrows.

The locus $\dot{k} \equiv dk/dt = 0$ as well as the dynamics of k, depicted by horizontal arrows, derive from (10). Steady states, denoted by A and C along the horizontal arm, and B along the vertical one, are identified by intersections between loci.

Figure 1: Capital accumulation for production under social planning



It is worth noting that the situation illustrated in figure 1 is only one out of five possible configurations, due to the fact that the position of the vertical line $f'(k) = \rho + \delta$ is independent of demand parameters, while the horizontal locus $q^{sp} = 2\sqrt{s_0}$ shifts upwards (downwards) as s increases. Therefore, we obtain one out of five possible regimes:

[1]. There exist three steady state points, with $k_A^{sp} < k_B^{sp} < k_C^{sp}$, (this is the situation depicted in figure 1).

[2]. There exist two steady state points, with $k_A^{sp} = k_B^{sp} < k_C^{sp}$.

[3]. There exist three steady state points, with $k_B^{sp} < k_A^{sp} < k_C^{sp}$.

[4]. There exist two steady state points, with $k_B^{sp} < k_A^{sp} = k_C^{sp}$.

[5]. There exists a unique steady state point, corresponding to B.

An intuitive explanation for the above taxonomy can be provided in the following terms. The vertical locus $f'(k) = \rho + \delta$ identifies a constraint on optimal capital embodying the monopolist's intertemporal preferences. Accordingly, the maximum steady state output would be that corresponding to (i) $\rho = 0$; and (ii) a capacity such that $f'(k) = \delta$. Yet, a positive discounting (that is, impatience) induces the planner to install a smaller steady state capacity, much the same as it happens in the well known Ramsey model (Ramsey, 1928).[9] On these grounds, define this level of k as the optimal capital constraint, and label it as $\hat{k}$. When the reservation price $\hat{s}$ is very large, points A and C either do not exist (regime [5]) or fall to the right of B (regimes [2], [3], and [4]). Under these circumstances, the capital constraint is operative and the planner chooses the capital accumulation corresponding to B. As we will see below, this is fully consistent with the dynamic properties of the steady state points.

Notice that, since both steady state points located along the horizontal locus entail the same levels of sales. As a consequence, point C is surely inefficient in that it requires a higher amount of capital. In point A, $dSW(t)/dq_i(t) = 0$; that is, the marginal instantaneous social welfare is nil.[10]

Now we come to the stability analysis of the above system. The joint dynamics of a (or q) and k, can be described by linearising (18) and (10) around (k^{sp}, a^{sp}), to get what follows:

$$\begin{bmatrix} \dot{k} \\ \\ \dot{a} \end{bmatrix} = \Xi \begin{bmatrix} (k - k^{sp}) \\ \\ (a - a^{sp}) \end{bmatrix} \tag{22}$$

[9] For a detailed exposition of the Ramsey model, I refer the reader to Blanchard and Fischer (1989, ch. 2).

[10] Point A corresponds to the optimal quantity emerging from the static version of the model (see Bonanno, 1987; Lambertini, 1995).

where

$$\Xi = \begin{bmatrix} f'(k) - \delta & -2 \\ \dfrac{(s_0 - 1/4 + a(1-a))}{2a-1} f''(k) & \dfrac{(4a(1-a) - 4s_0 - 1)}{2(2a-1)^2}(f'(k) - \rho - \delta) \end{bmatrix}$$

The stability properties of the system in the neighbourhood of the steady state depend upon the trace and determinant of the 2×2 matrix Ξ. In studying the system, we confine to steady state points. The trace of Ξ is

$$tr(\Xi) = f'(k) - \delta + \frac{(4a(1-a) - 4s_0 - 1)}{2(2a-1)^2}(f'(k) - \rho - \delta) \qquad (23)$$

yielding $tr(\Xi) = \rho > 0$ in correspondence of both $a = 1/2 + \sqrt{s_0}$ and $f'(k) = \rho + \delta$: Joint with the evaluation of the determinant $\Delta(\Xi)$ at the same points, the following taxonomy obtains.

Regime [1]. In A, $\Delta(\Xi) < 0$; hence this is a saddle point. In B; $\Delta(\Xi) > 0$; so that B is an unstable focus. In C, $\Delta(\Xi) < 0$; and this is again a saddle point, with the horizontal line as the stable arm.

Regime [2]. In this regime, A coincides with B; so that we have only two steady states which are both are saddle points. In A = B, the saddle path approaches the saddle point from the left only, while in C the stable arm is again the horizontal line.

Regime [3]. Here, B is a saddle; A is an unstable focus; C is a saddle point, as in regimes [1] and [2].

Regime [4]. Here, points A and C coincide. B remains a saddle, while A = C is a saddle whose converging arm proceeds from the right along the horizontal line.

Regime [5]. Here, there exists a unique steady state point, B; which is also a saddle point.

We can sum up the above discussion as follows. The unique efficient and non-unstable steady state point is B if $k_B^{sp} \equiv \hat{k} < k_A$; while it is A if the opposite inequality holds. Such a point is always a saddle. Individual equilibrium output is $q^{sp} = 2\sqrt{s_0}$ if the equilibrium is identified by point A; or the level corresponding to the optimal capital constraint $\hat{k}$ if the equilibrium is identified by point B. The reason is that, if the capacity at which marginal instantaneous profit is nil is larger than the optimal capital constraint, the latter becomes binding. Otherwise, the capital constraint is irrelevant, and the planner's decisions in each period are solely driven by the unconstrained maximisation of instantaneous social welfare.

The above discussion can be summarised as follows:

Proposition 1 If $k_B^{sp} \equiv \hat{k} > k_A$; the steady state output level is

- $q^{sp} = 2\sqrt{s_0}$ iff $s_0 \in \left[0, \frac{1}{4}\right)$; and partial market coverage obtains;
- $q^{sp} = 1$ iff $s_0 \geq \frac{1}{4}$; and full market coverage obtains.

If $k_B^{sp} \equiv \hat{k} < k_A$; the steady state output is $q^{sp} = f\left(\hat{k}\right)$, and

- partial market coverage obtains (i) for all $s_0 \in \left[0, \frac{1}{4}\right)$; or (ii) for all $s_0 \geq \frac{1}{4}$; iff $f\left(\hat{k}\right) < 1$.
- full market coverage obtains iff $s_0 \geq \frac{1}{4}$ and $f\left(\hat{k}\right) \geq 1$.

3.2 Capital accumulation in a profit-seeking monopoly

The objective of the monopolist is

$$\max_{a(t)} \int_0^\infty e^{-\rho t} R(t)dt = \int_0^\infty e^{-\rho t} \left[s_0 - \left(\frac{1}{2} - a(t)\right)^2 \right] [2a(t) - 1]\, dt \quad (24)$$

$$s.t.\ \frac{\partial k(t)}{\partial t} = f(k(t)) - q(t) - \delta k(t) \quad (25)$$

where ρ denotes the same time discounting as for the planner. The corresponding Hamiltonian function is:

$$H(t) = e^{-\rho t} \cdot \left\{ \left[s_0 - \left(\frac{1}{2} - a(t) \right)^2 \right] [2a(t) - 1] + \lambda(t) [f(k(t)) - 2a(t) + 1 - \delta k(t)] \right\}; \tag{26}$$

where, again, $\lambda(t) = \beta(t)e^{\rho t}$; and $\beta(t)$ is the co-state variable associated to $k(t)$.

The solution to the monopolist's problem is largely analogous to that of the planner as illustrated in section 3.1. Therefore, detailed calculations are in Appendix 1. However, one specific result is worth stating here:

Lemma 2 Under monopoly, the shadow price of capital is

$$\lambda(t) = 3a(t)[1 - a(t)] + s_0 - 3/4$$

which is lower than the market price $p(t) = s(t) - [\theta(t) - a(t)]^2$ evaluated at $\theta(t)$; for all admissible $a(t) \neq 1/2$. At $a(t) = 1/2$; we have $\lambda(t) = p(t)$.

According to Lemma 2, the value attached by the monopolist to a current unit of sales is larger than the shadow value characterising a further unit of capital which would increase productive capacity in the future. Since any increase in productive capacity requires some unsold output, Lemma 2 says that we should expect to observe cases where the monopolist undersupplies the market in steady state, as compared to the planner.

The steady state output is either $q^m(t) = 2\sqrt{s_0/3}$, where superscript m stands for monopoly, or the quantity corresponding to a capacity $\hat{k} = f'^{-1}(\rho + \delta)$ (i.e., the Ramsey equilibrium is obviously the same as under social planning).

The main results can be stated as follows:

Proposition 2 If $\hat{k} > k_A^m$; the steady state output level is

- $q^m = 2\sqrt{s_0/3} \Leftrightarrow s_0 \in \left[0, \frac{3}{4}\right)$; and partial market coverage obtains;

- $q^m = 1$ iff $\hat{s} \geq \frac{3}{4}$; and full market coverage obtains.

If $\hat{k} < k_A^m$; the steady state output is $q^m = f\left(\hat{k}\right)$, where $\hat{k}$ is the capital level at which $f'(k) = \rho + \delta$; and

- partial market coverage obtains (i) for all $s_0 \in \left(0; \frac{3}{4}\right)$; or (ii) for all $s_0 \geq \frac{3}{4}$; iff $f\left(\hat{k}\right) < 1$:
- full market coverage obtains iff $s_0 \geq \frac{3}{4}$ and $f\left(\hat{k}\right) \geq 1$:

Propositions 1 and 2 produce the following:

Corollary 1 Suppose $\hat{k} > k_A$ under both regimes. If so, then $q^m < q^{sp} = 1$ for all $s_0 \in \left(\frac{1}{4}; \frac{3}{4}\right)$; i.e., the planner covers the whole market while the monopolist does not.

Corollary 2 Suppose $\hat{k} \leq k_A$ under both regimes. If so, then equilibrium output is $q^m = q^{sp} = f\left(\hat{k}\right)$.

The intuition behind the output distortion characterising monopoly and highlighted in Corollary 1 is provided by Lemmata 1-2, which stress the difference between the unit prices of output and capital in the two regimes. In the situation considered in Corollary 2, social welfare is the same under the two regimes, except that the distribution of total surplus is different due to the different pricing policies adopted, i.e., monopoly pricing vs marginal cost pricing.

4 Advertising

Here, I abstract from the problem of capital accumulation for production and focus upon the investment in advertising aimed at increasing consumers' reservation price s; according to the dynamic equation (8). For the problem to be economically meaningful, I assume that the initial state is $s(0) = s_0$; with $s_0 \in \left(0; \frac{1}{4}\right)$ under planning and $s_0 \in \left(0; \frac{3}{4}\right)$ under monopoly.

4.1 Advertising under social planning

In scenario 2, the planner's problem consists in

$$\max_{a(t);x(t)} \int_0^\infty e^{-\rho t}\{SW(t) - \rho x(t)\}\,dt = \tag{27}$$

$$= \int_0^\infty e^{-\rho t}\left\{\frac{(2a(t)-1)\,[12s(t)-1+4a(t)\,(1-a(t))]}{12} - \rho x(t)\right\}dt$$

$$\text{s.t.:}\quad \frac{\partial s(t)}{\partial t} = b\sqrt{x(t)} - \delta s(t) \tag{28}$$

The relevant Hamiltonian is

$$\begin{aligned} H(t) \;=\; & e^{-\rho t}\cdot\left\{\frac{(2a(t)-1)\,[12s(t)-1+4a(t)\,(1-a(t))]}{12} - \rho x(t) + \right. \\ & \left. +\,\mu(t)\left[b\sqrt{x(t)} - \delta s(t)\right]\right\} \end{aligned} \tag{29}$$

where (i) I assume that the unit rental price of the capital to be invested in the advertising coincides with the discount rate; (ii) δ is the constant depreciation and $\mu(t) = \lambda(t)e^{\rho t}$; and $\lambda(t)$ is the co-state variable associated to $s(t)$. The first order conditions are:

$$\frac{\partial H(t)}{\partial a(t)} = -2\,[a(t)]^2 + 2a(t) + 2s(t) - \frac{1}{2} = 0; \tag{30}$$

$$\frac{\partial H(t)}{\partial x(t)} = -\rho + \frac{\mu(t)b}{2\sqrt{x(t)}} = 0; \tag{31}$$

$$-\frac{\partial H(t)}{\partial s(t)} = \frac{\partial \lambda(t)}{\partial t} \;\Rightarrow\; \frac{\partial \mu(t)}{\partial t} = 1 - 2a(t) + \mu(t)\,(\rho+\delta)\,; \tag{32}$$

The transversality condition is:

$$\lim_{t\to\infty} \lambda(t)\cdot x(t) = 0. \tag{33}$$

From (30) I obtain:

$$a(t) = \frac{1}{2} + \sqrt{s(t)} \tag{34}$$

while from 30) I obtain:

$$\mu(t) = \frac{2\rho\sqrt{x(t)}}{b} \tag{35}$$

and

$$x(t) = \left[\frac{\mu(t)b}{2\rho}\right]^2; \tag{36}$$

which can be differentiated w.r.t. time to get

$$\frac{dx(t)}{dt} = \frac{b^2}{2\rho^2}\mu(t)\frac{d\mu(t)}{dt}. \tag{37}$$

Therefore,

$$\text{sign}\left\{\frac{dx(t)}{dt}\right\} = \text{sign}\left\{\mu(t)\frac{d\mu(t)}{dt}\right\}. \tag{38}$$

Using (32), (34) and (35), (37) rewrites as:

$$\frac{dx(t)}{dt} = \frac{2\sqrt{x(t)}\left[\rho(\rho+\delta)\sqrt{x(t)} - b\sqrt{s}\right]}{\rho}; \tag{39}$$

which, together with (28), fully describes the dynamic properties of the model. Solving (39) w.r.t. $x(t)$ yields the optimal investment in advertising as a function of $s(t)$:

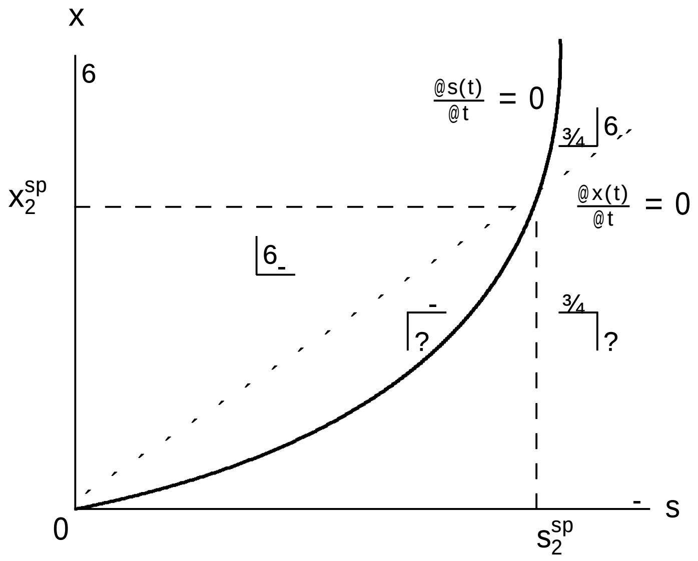
$$x_1(t) = 0\,;\; x_2(t) = \frac{b^2 s(t)}{\rho^2(\rho+\delta)^2} \tag{40}$$

while the locus $\frac{\partial s(t)}{\partial t} = 0$ is given by $s(t) = \frac{b}{\delta}\sqrt{x(t)}$. Therefore, long-run equilibrium points are:

$$\{s_1^{sp} = 0\,;\, x_1^{sp} = 0\}\,;\; \left\{s_2^{sp} = \frac{b^4}{[\delta\rho(\rho+\delta)]^2}\,;\, x_2^{sp} = \left[\frac{b^6}{\delta\rho^2(\rho+\delta)^2}\right]^2\right\}. \tag{41}$$

The phase diagram is illustrated in figure 2.

Figure 2 : Advertising under social planning



As to the stability analysis of the system, the description of the joint dynamics of s and x obtains by linearising (28) and (39) around $(s_1^{sp}; x_1^{sp})$ and $(s_2^{sp}; x_2^{sp})$; to get:

$$\begin{bmatrix} \dot{s} \\ \dot{x} \end{bmatrix} = \Omega \begin{bmatrix} (s - s^{sp}) \\ (x - x^{sp}) \end{bmatrix} \tag{42}$$

with

- $tr(\Omega) = 2\rho + \delta > 0$ and $\Delta(\Omega) = -2\delta(\rho + \delta) < 0$ in $(s_1^{sp}; x_1^{sp})$.[11]
- $tr(\Omega) = \rho > 0$ and $\Delta(\Omega) = -\frac{\delta(\rho+\delta)}{2} < 0$ in $(s_2^{sp}; x_2^{sp})$.

In $(s_1^{sp}; x_1^{sp})$ the discounted payoff is obviously nil, and second order con-

[11] The trace and determinant of Ω in $(s_1^{sp}; x_1^{sp})$ have been calculated using de l'Hôpital's rule (see Appendix 2).

ditions reveal that it is a minimum point. In $(s_2^{sp}; x_2^{sp})$; we have

$$a(s_2^{sp}) < 1 \text{ for all } b < \sqrt{\frac{\delta\rho(\rho+\delta)}{2}}: \tag{43}$$

The equilibrium price is zero. The above results can be summarised as follows:

Proposition 3 For all $b < \sqrt{\frac{\delta\rho(\rho+\delta)}{2}}$; partial market coverage obtains at the steady state equilibrium where $s^{sp} = \frac{b^4}{[\delta\rho(\rho+\delta)]^2}$ and $x^{sp} = \frac{b^6}{\left[\delta\rho^2(\rho+\delta)^2\right]^2}$:

For all $b \geq \sqrt{\frac{\delta\rho(\rho+\delta)}{2}}$; full market coverage obtains at the steady state equilibrium where $s^{sp} = \frac{1}{4}$ and $x^{sp} = \frac{b^2}{4\rho^2(\rho+\delta)^2}$:

4.2 Advertising under monopoly

The problem of optimal intertemporal investment in advertising under monopoly can be quickly dealt with, as it is largely analogous to the planner's problem, except that the monopolist maximises the net discounted profit flow:

$$\max_{a(t);x(t)} \int_0^\infty e^{-\rho t}\{R(t) - \rho x(t)\}\,dt = \tag{44}$$

$$= \int_0^\infty e^{-\rho t}\left\{(2a(t)-1)\left[s(t) - (a(t)-1/2)^2\right] - \rho x(t)\right\}dt$$

$$\text{s.t.:} \quad \frac{\partial s(t)}{\partial t} = b\sqrt{x(t)} - \delta s(t) \tag{45}$$

The relevant Hamiltonian is

$$\begin{aligned} H(t) \;=\; & e^{-\rho t}\cdot\left\{(2a(t)-1)\left[s(t) - (a(t)-1/2)^2\right] - \rho x(t) + \right. \\ & \left. + \mu(t)\left(b\sqrt{x(t)} - \delta s(t)\right)\right\}: \end{aligned} \tag{46}$$

The first order conditions are:

$$\frac{\partial H(t)}{\partial a(t)} = -6[a(t)]^2 + 6a(t) + 2s(t) - \frac{3}{2} = 0 \;\Rightarrow\; a(t) = \frac{1}{2} + \sqrt{\frac{s(t)}{3}}; \tag{47}$$

$$\frac{\partial H(t)}{\partial x(t)} = -\rho + \frac{\mu(t)b}{2\sqrt{x(t)}} = 0\,; \tag{48}$$

$$-\frac{\partial H(t)}{\partial s(t)} = \frac{\partial \varpi(t)}{\partial t} \Rightarrow \frac{\partial \mu(t)}{\partial t} = 1 - 2a(t) + \mu(t)(\rho + \delta)\,; \tag{49}$$

The transversality condition is:

$$\lim_{t \to \infty} \varpi(t) \cdot x(t) = 0\,. \tag{50}$$

The dynamic behaviour of advertising is described by:

$$\frac{dx(t)}{dt} = \frac{2\sqrt{x(t)}\left[3\rho(\rho+\delta)\sqrt{x(t)} - b\sqrt{3s}\right]}{3\rho} \tag{51}$$

yielding

$$x^m = \frac{b^2 s}{3\rho^2(\rho+\delta)^2}\,. \tag{52}$$

This allows me to formulate

Lemma 3 Given $s < 3/4$, the monopolist's advertising effort is one third of the socially optimal investment.

The system $\{ds/dt = 0\,;\ dx/dt = 0\}$ has the following critical points:

$$\begin{aligned} s_1^m &= 0\,;\ x_1^m = 0 \\ s_2^m &= \frac{b^4}{3\left[\delta\rho(\rho+\delta)\right]^2}\,;\ x_1^m = \left[\frac{b^6}{3\delta\rho^2(\rho+\delta)^2}\right]^2 \end{aligned} \tag{53}$$

Therefore, I can state

Proposition 4 For all $b < \sqrt{\frac{3\delta\rho(\rho+\delta)}{2}}$, partial market coverage obtains at the steady state equilibrium where $s^m = \frac{b^4}{3\left[\delta\rho(\rho+\delta)\right]^2}$ and $x^m = \left[\frac{b^6}{3\delta\rho^2(\rho+\delta)^2}\right]^2$.

For all $b \geq \sqrt{\frac{3\delta\rho(\rho+\delta)}{2}}$, full market coverage obtains at the steady state equilibrium where $s^m = \frac{3}{4}$ and $x^m = \frac{b^2}{4\rho^2(\rho+\delta)^2}$.

The underinvestment characterising the monopoly optimum becomes evident when full market coverage is reached at $s^m = 3/4$. In that situation, the steady state advertising effort equals the investment carried out by the monopolist to obtain the same coverage at $s^{sp} = 1/4$. Propositions 3 and 4 entail the following Corollary:

Corollary 3 For all $b < \sqrt{\frac{3\delta\rho(\rho+\delta)}{2}}$; the monopolist distorts both output and advertising as compared to the social optimum.

5 Capital accumulation and advertising

The foregoing sections evaluated optimal production and advertising decisions separately, to build up a benchmark for a more realistic setting where they are allowed to interact with each other. I am now in a position to illustrate the joint dynamics of capacity accumulation and advertising investment.

5.1 Capital accumulation and advertising under social planning

Now the planner's problem consists in

$$\max_{a(t);s(t)} \int_0^\infty e^{-\rho t} \{SW(t) - \rho x(t)\} dt = \tag{54}$$

$$= \int_0^\infty e^{-\rho t} \left\{ \frac{(2a(t)-1)[12s(t)-1+4a(t)(1-a(t))]}{12} - \rho x(t) \right\} dt$$

$$s.t.: \quad \frac{\partial k(t)}{\partial t} = f(k(t)) - q(t) - \delta k(t) \tag{55}$$

$$\frac{\partial s(t)}{\partial t} = b\sqrt{x(t)} - \delta s(t) \tag{56}$$

where $q(t) = 2a(t) - 1$. The Hamiltonian is

$$H(t) = e^{-\rho t} \cdot \left\{ \frac{(2a(t) - 1)\,[12s(t) - 1 + 4a(t)\,(1 - a(t))]}{12} - \rho x(t) + \right. \tag{57}$$
$$\left. + \lambda(t)\,[f(k(t)) - 2a(t) + 1 - \delta k(t)] + \mu(t)\,\left[b\sqrt{x(t)} - \delta s(t) \right] \right\}$$

where $\lambda(t) = \beta(t)e^{\rho t}$ and $\mu(t) = \varpi(t)e^{\rho t}$, $\beta(t)$ and $\varpi(t)$ being the co-state variables associated to $k(t)$ and $x(t)$, respectively. The following first order and transversality conditions must hold:

$$\frac{\partial H(t)}{\partial a(t)} = -2\left\{ [a(t)]^2 - a(t) - s(t) + \lambda(t) \right\} - \frac{1}{2} = 0; \tag{58}$$

$$\frac{\partial H(t)}{\partial x(t)} = -\rho + \frac{\mu(t)b}{2\sqrt{x(t)}} = 0; \tag{59}$$

$$-\frac{\partial H(t)}{\partial k(t)} = \frac{\partial \beta(t)}{\partial t} \Rightarrow \frac{\partial \lambda(t)}{\partial t} = [\rho + \delta - f'(k(t))]\,\lambda(t); \tag{60}$$

$$-\frac{\partial H(t)}{\partial s(t)} = \frac{\partial \varpi(t)}{\partial t} \Rightarrow \frac{\partial \mu(t)}{\partial t} = 1 - 2a(t) + \mu(t)\,(\rho + \delta); \tag{61}$$

$$\lim_{t \to \infty} \beta(t) \cdot k(t) = 0; \tag{62}$$

$$\lim_{t \to \infty} \varpi(t) \cdot x(t) = 0. \tag{63}$$

From (58), I obtain $a(t) = 1/2 + \sqrt{s(t) - \lambda(t)}$ and $\lambda(t) = a(t)\,[1 - a(t)] + s(t) - 1/4$. Then, from (59), $\mu(t) = 2\rho\sqrt{x(t)}/b$ and $x(t) = [b\mu(t)/(2\rho)]^2$, which implies

$$\text{sign}\left\{\frac{dx(t)}{dt}\right\} = \text{sign}\left\{\mu(t)\frac{d\mu(t)}{dt}\right\}; \tag{64}$$

where, from (59-61), I can plug the expression for $\mu(t)$ and $d\mu(t)/dt$. Moreover,

$$\text{sign}\left\{\frac{da(t)}{dt}\right\} = \text{sign}\left\{\frac{ds(t)}{dt} - \frac{d\lambda(t)}{dt}\right\}. \tag{65}$$

Substituting and simplifying, I obtain:

$$\text{sign}\left\{\frac{dx(t)}{dt}\right\} = \text{sign}\left\{\frac{2\rho\sqrt{x(t)}\left[2\rho\,(\rho + \delta)\sqrt{x(t)} - b\,(2a(t) - 1)\right]}{b^2}\right\} \tag{66}$$

$$\operatorname{sign}\left\{\frac{da(t)}{dt}\right\} = \tag{67}$$

$$= \operatorname{sign}\left\{ b\sqrt{x(t)} - \delta s(t) - [\rho + \delta - f'(k(t))]\left[s(t) - \frac{1}{4} + a(t) - (a(t))^2 \right] \right\} .$$

Together with (55) and (56), conditions (66-67) describes the dynamic features of the model. Although the system cannot be given a graphical illustration,[12] the steady state equilibria can be characterised analytically. First, I obtain the equilibrium expressions for x and s as a function of a :

$$x^{sp} = \left[\frac{b(2a-1)}{2\rho(\rho+\delta)}\right]^2 = \left[\frac{bq}{2\rho(\rho+\delta)}\right]^2 ; \; s^{sp} = \frac{b^2(2a-1)}{2\delta\rho(\rho+\delta)} = \frac{b^2 q}{2\delta\rho(\rho+\delta)} . \tag{68}$$

Now, we have that

$$\operatorname{sign}\left\{\frac{da}{dt}\right\} = \operatorname{sign}\left\{(1-2a)\left[2b^2 + \delta\rho(\rho+\delta)(1-2a)\right][\rho+\delta-f'(k)]\right\} \tag{69}$$

yielding

$$a_1^{sp} = \frac{1}{2} ; \; a_2^{sp} = \frac{2b^2 + \delta\rho(\rho+\delta)}{2\delta\rho(\rho+\delta)} ; \; f'(k) = \rho + \delta . \tag{70}$$

In a_1^{sp}; we have a minimum point where x and s are both nil. In $f'(k) = \rho + \delta$; we are in the Ramsey equilibrium dictated by intertemporal capacity accumulation alone, with $q = f(\hat{k})$. In a_2^{sp}; we have a saddle point where

$$x^{sp} = \frac{b^6}{\left[\delta\rho^2(\rho+\delta)^2\right]^2} ; \; s^{sp} = \frac{b^4}{[\delta\rho(\rho+\delta)]^2} ; \tag{71}$$

which coincide with the equilibrium values derived in section 4.1.

The above discussion leads to

Proposition 5 If $f(\hat{k}) > 2a^{sp} - 1$ and $b < \sqrt{\frac{\delta\rho(\rho+\delta)}{2}}$; the steady state equilibrium under social planning is a saddle point with partial market coverage.

[12] However, given a pair $(s; x)$; the dynamic properties of production are described by the analogous to figure 1. Likewise, given $(a; k)$; the dynamic properties of advertising are as in figure 2.

If $f\left(\hat{k}\right) > 2a^{sp} - 1$ and $b \geq \sqrt{\frac{\delta\rho(\rho+\delta)}{2}}$, the steady state equilibrium under social planning is a saddle point with full market coverage.

5.2 Capital accumulation and advertising in a profit-seeking monopoly

Since the monopoly problem is largely analogous to the planner's, except for the instantaneous (gross) payoff, here I confine to the exposition of the maximum problem without giving the proof of the results. The monopolist's problem is defined as follow:

$$\max_{a(t),s(t)} \int_0^{\infty} e^{-\rho t} \{R(t) - \rho x(t)\}\, dt = \tag{72}$$

$$= \int_0^{\infty} e^{-\rho t} \left\{ (2a(t) - 1)\left[s(t) - (a(t) - 1/2)^2 \right] - \rho x(t) \right\} dt$$

$$\text{s.t.} \quad \frac{\partial k(t)}{\partial t} = f(k(t)) - q(t) - \delta k(t) \tag{73}$$

$$\frac{\partial s(t)}{\partial t} = b\sqrt{x(t)} - \delta s(t) \tag{74}$$

where $q(t) = 2a(t) - 1$. The corresponding Hamiltonian is

$$\begin{aligned} H(t) = e^{-\rho t} \cdot \Big\{ & (2a(t) - 1)\left[s(t) - (a(t) - 1/2)^2 \right] - \rho x(t) + \\ & + \lambda(t)\left[f(k(t)) - 2a(t) + 1 - \delta k(t) \right] + \mu(t)\left(b\sqrt{x(t)} - \delta s(t) \right) \Big\}. \end{aligned} \tag{75}$$

As in the case of social planning, the equilibrium expressions for x and s as a function of a are:

$$x^m = \left[\frac{b(2a-1)}{2\rho(\rho+\delta)} \right]^2 = \left[\frac{bq}{2\rho(\rho+\delta)} \right]^2 ; \; s^m = \frac{b^2(2a-1)}{2\delta\rho(\rho+\delta)} = \frac{b^2 q}{2\delta\rho(\rho+\delta)}. \tag{76}$$

In steady state, we have

$$a^m = \frac{2b^2 + 3\delta\rho(\rho+\delta)}{6\delta\rho(\rho+\delta)} ; \; f'(k) = \rho + \delta \tag{77}$$

and, at $a = a^m$:

$$x^m = \frac{b^6}{\left[3\delta\rho^2(\rho+\delta)^2\right]^2}\,;\; s^m = \frac{b^4}{3\left[\delta\rho(\rho+\delta)\right]^2}\,: \tag{78}$$

In summary,

Proposition 6 *If $f\left(\frac{b}{k}\right) > 2a^m - 1$ and $b < \sqrt{\frac{3\delta\rho(\rho+\delta)}{2}}$, the steady state equilibrium under social planning is a saddle point with partial market coverage.*

If $f\left(\frac{b}{k}\right) > 2a^m - 1$ and $b \geq \sqrt{\frac{3\delta\rho(\rho+\delta)}{2}}$, the steady state equilibrium under social planning is a saddle point with full market coverage.

Finally, Propositions 5 and 6 produce the following:

Corollary 4 *Suppose $f\left(\frac{b}{k}\right) > 2a^{sp} - 1$. Then, for all*

$$b \in \left[\sqrt{\frac{\delta\rho(\rho+\delta)}{2}}\,;\; \sqrt{\frac{3\delta\rho(\rho+\delta)}{2}}\right);$$

full market coverage obtains under social planning while partial market coverage obtains under monopoly, with the monopolists investing less than the planner in both capacity and advertising.

That is, as from Propositions 1 and 2, the well known output distortion associated to monopoly pricing translates into a lower steady state capacity as compared to the social optimum. Moreover, from Propositions 3 and 4, we know that under partial market coverage the monopolist underinvests in advertising. Therefore, when the firm invests to accumulate capacity and increase consumers' reservation price, the profit-seeking monopoly is affected by distortions along both dimensions of the investment portfolio.

Of course there are many other possible configurations, one of which is of some interest. When $f\left(\frac{b}{k}\right) \leq 2a^m - 1$, then the firm produces $q^* =$

$\min\left\{f\left(\widehat{k}\right), 1\right\}$ under both regimes, and

$$x = \left[\frac{b\min\left\{f\left(\widehat{k}\right), 1\right\}}{2\rho\left(\rho+\delta\right)}\right]^2 ; \; s = \frac{b^2\min\left\{f\left(\widehat{k}\right), 1\right\}}{2\delta\rho\left(\rho+\delta\right)} . \tag{79}$$

Accordingly, we have our the final result:

Proposition 7 *If $f\left(\widehat{k}\right) \leq 2a^m - 1$, then the Ramsey equilibrium obtains irrespective of the firm's objective, and no distortion affects the profit-seeking monopoly as compared to social planning.*

Obviously, the above claim abstracts from any considerations concerning the distribution of surplus, which is affected by the different pricing policy adopted by the firm in the two regimes. However, it highlights a property that was necessarily disregarded by the static analysis of the same model (Bonanno, 1987; and Lambertini, 1995), and suggests interesting implications for industrial policy.

Consider the situation where $f\left(\widehat{k}\right) \in \left(2a^m - 1, 2a^{sp} - 1\right)$. In this case, the monopoly steady state is driven by demand conditions, while the planner's steady state is the Ramsey equilibrium. A policy maker could design an incentive, such as a subsidy to production (i.e., a subsidy to capacity accumulation) that might induce the profit-seeking monopolist to expand sales so as to reduce, if not eliminate at all, the output distortion. This would also exert beneficial effects on the monopolist's steady state advertising effort.

6 Concluding remarks

I have investigated the optimal capacity accumulation and advertising investment of a single-product firm operating in a spatial market with a uniform consumer distribution, comparing the steady state behaviour of a profit-seeking monopolist versus that of a benevolent social planner.

There emerges that the monopolist distorts both capital accumulation (and sales) and advertising, whenever partial market coverage obtains at equilibrium. This distortions disappear when the steady state equilibrium is dictated only by the conditions driving intertemporal capital accumulation. Therefore, a subsidy could be introduced, so as to drive the monopolist towards the Ramsey equilibrium.

Appendices

Appendix 1. Capital accumulation under monopoly

The monopolist's problem outlined in section 3.2 is solved as follows. From (26), the necessary and sufficient conditions for a path to be optimal are:

$$\frac{\partial H(t)}{\partial a(t)} = -6\left[a(t)\right]^2 + 6a(t) + 2s_0 - \frac{3}{2} - 2\lambda(t) = 0; \tag{a1}$$

$$-\frac{\partial H(t)}{\partial k(t)} = \frac{\partial \beta(t)}{\partial t} \Rightarrow \frac{\partial \lambda(t)}{\partial t} = \left[\rho + \delta - f'(k(t))\right]\lambda(t); \tag{a2}$$

$$\lim_{t \to \infty} \beta(t) \cdot k(t) = 0. \tag{a3}$$

From (a1), I obtain

$$a(t) = \frac{1}{2} + \sqrt{\frac{s_0 - \lambda(t)}{3}} \tag{a4}$$

which yields $\lambda(t) = 3a(t)\left[1 - a(t)\right] + s_0 - 3/4$. This, in combination with (2) and $\ell(t) = 1/2$, proves Lemma 2. Moreover, (a4) can be differentiated w.r.t. time to get

$$\frac{da(t)}{dt} = \frac{-d\lambda(t)/dt}{2\sqrt{3\left[s_0 - \lambda(t)\right]}}. \tag{a5}$$

Thanks to (a2), the expression in (a5) simplifies as follows:

$$\frac{da(t)}{dt} = -\frac{\left[\rho + \delta - f'(k(t))\right]\lambda(t)}{2\sqrt{3\left[s_0 - \lambda(t)\right]}}. \tag{a6}$$

Again from (a1), $\lambda(t) = 3a(t)\left[1 - a(t)\right] + s_0 - 3/4$, and (a6) rewrites as follows:

$$\frac{da(t)}{dt} = -\frac{\left[3a(t) - 3\left(a(t)\right)^2 + s_0 - 3/4\right]\left[\rho + \delta - f'(k(t))\right]}{2\sqrt{3\left[3/4 - 3a(t) + 3\left(a(t)\right)^2\right]}}. \tag{a7}$$

Hence, we have that

$$\text{sign}\left\{\frac{da(t)}{dt}\right\} = \text{sign}\left\{\left[3a(t) - 3\left(a(t)\right)^2 + s_0 - 3/4\right]\left[f'(k(t)) - \rho - \delta\right]\right\}. \tag{a8}$$

The expression on the r.h.s. of (a8) is zero when

$$f'(k(t)) = \rho + \delta; \tag{a9}$$

$$a(t) = \frac{1}{2} \pm \sqrt{\frac{s_0}{3}}: \tag{a10}$$

The critical point (a9) denotes the situation where the marginal product of capital is just sufficient to cover discounting and depreciation. The smaller solution in (a10) can be disregarded on the basis of the assumption that $a(t) \in (1/2; 1]:$ Therefore, the long run equilibrium output is either $q^m(t) = 2\sqrt{s_0/3}$, where superscript m stands for monopoly, or the quantity corresponding to a capacity $f'^{-1}(\rho + \delta):$ Observe that the Ramsey equilibrium is the same as under social planning. It is also worth noting that $q^m(t) \le 1$ for all $s_0 \le 3/4$: At $q^m(t) = 2\sqrt{s_0/3}$, the optimal price is $p^m(t) = 2s_0/3$:

The discussion and the graphical illustration of steady states are omitted in that they are analogous to those proposed above for the case of social planning, except that the demand-driven long run equilibrium at $q^m(t) = 2\sqrt{s_0/3}$ obviously involves a smaller quantity (and a higher price) than observed at the corresponding equilibrium under planning.

Appendix 2. Advertising under social planning: stability analysis

The dynamic properties of the planner's problem described in section 4.1 are summarised by the following matrix:

$$\Omega = \begin{bmatrix} -\delta & \dfrac{b}{2\sqrt{x}} \\ -\dfrac{b}{\rho}\sqrt{\dfrac{x}{s}} & \dfrac{2\rho\sqrt{x}(\rho+\delta) - b\sqrt{s}}{\rho\sqrt{x}} \end{bmatrix}$$

with

$$tr(\Omega) = \frac{\rho\sqrt{x}(2\rho+\delta) - b\sqrt{s}}{\rho\sqrt{x}} \tag{a11}$$

$$\Delta(\Omega) = \frac{\sqrt{x}\,[b^2 - 4\rho\delta(\rho+\delta)\sqrt{s}] + 2b\delta s}{2\rho\sqrt{sx}} \tag{a12}$$

Consider first the point $(s_1^{sp} = 0; x_1^{sp} = 0)$; and examine the trace. When $s = 0$; we have $\operatorname{tr}(\Psi) = 2\rho + \delta$; independently of x:

Now consider the determinant. In $(0;0)$; $\Delta(\Psi)$ is indeterminate. Hence, I apply de l'Hôpital's rule by calculating the limits of the numerator and denominator of $\Delta(\Psi)$ as s tends to zero. This yields:

$$\tilde{\Delta}(\Psi) = \frac{2\delta\left[b\sqrt{s} - \rho(\rho + \delta)\sqrt{x}\right]}{\rho\sqrt{x}} \tag{a13}$$

which, again, is indeterminate. Therefore, I apply once again de l'Hôpital's rule by calculating the limits of the numerator and denominator of $\Delta(\Psi)$ as x tends to zero, obtaining:

$$\overline{\Delta}(\Psi) = -2\delta(\rho + \delta) : \tag{a14}$$

Finally, calculating $\operatorname{tr}(\Psi)$ and $\Delta(\Psi)$ in $(s_2^{sp}; x_2^{sp})$ is a matter of simple algebra.

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